

COMPLEMENTARY RAMSEY NUMBERS

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ABSTRACT. In this paper, we propose a new generalization of Ramsey numbers which seems to be untreated in the literature. Instead of requiring the existence of a monochromatic clique, we consider the existence of a clique which avoids one of the colors in an edge coloring. These numbers are called complementary Ramsey numbers, and we derive their basic properties. We also establish their connections to graph factorizations.

1. INTRODUCTION

For any given positive integers n_1, \dots, n_c , there is a number, $\bar{R}(n_1, \dots, n_c)$, such that if the edges of a complete graph of order $\bar{R}(n_1, \dots, n_c)$ are colored with c different colors, then for some i between 1 and c , there exists a complete subgraph of order n_i all of whose edges have colors different from i . Note that $\bar{R}(n_1, n_2) = R(n_2, n_1)$, an ordinary Ramsey number.

Since K_4 admits a 1-factorization, that is, a partition of the edge-set into matchings, we see $\bar{R}(3, 3, 3) \geq 5$. More generally, the existence of l mutually orthogonal Latin squares of order n implies that $\bar{R}(n+1, n+1, \dots, n+1)$ (repeated $l+2$ times) is at least $n^2 + 1$. In this paper, we generalize and strengthen these arguments to establish basic properties of complementary Ramsey numbers. The main tools are the concept of graph factorizations and resolvable designs. In particular, we point out that the existence of complete set of mutually orthogonal Latin squares can be stated in terms of complementary Ramsey numbers.

If we could show $\bar{R}(5, 5, 5) = 12$, then this would have contributed to the proof of a conjecture of Einhorn–Schoenberg [11]. Although this was not the case, the approach uses the same idea as the definition of complementary Ramsey numbers, so we believe it is worthwhile to formulate it as a formal definition.

This paper is organized as follows. In Sect. 2, we give definitions and derive immediate consequences. In Sect. 3, we derive some inequalities for complementary Ramsey numbers. In Sect. 4, we point out connections to graph factorizations. In Sect. 5, we list some consequences obtained from the connections to graph factorizations. Finally, in Sect. 6, we tabulate complementary Ramsey numbers with small parameters.

2. DEFINITIONS AND NOTATION

For a positive integer n , we denote the set $\{1, \dots, n\}$ by $[n]$, and the set of all n -element subsets of a set X by $\binom{X}{n}$. For positive integers k, n , we denote the set of all edge-coloring of the complete graph K_n by k colors, by $C(n, k)$:

$$C(n, k) = \{f \mid f : \binom{[n]}{2} \rightarrow [k]\}.$$

If G is a graph, then we denote by $\alpha(G)$ the independence number of G , and by $\omega(G)$ the clique number of G . We identify a graph whose vertex set is $[n]$, with its set of edges. In particular, for $f \in C(n, k)$ and $i \in [k]$, $f^{-1}(i)$ is regarded as the graph $([n], f^{-1}(i))$. We use the abbreviations $\alpha_i(f) = \alpha(f^{-1}(i))$, $\omega_i(f) = \omega(f^{-1}(i))$. The Ramsey number can be defined as follows:

$$R(m_1, \dots, m_k) = \min\{n \in \mathbb{N} \mid \forall f \in C(n, k), \exists i \in [k], \omega_i(f) \geq m_i\},$$

where m_1, \dots, m_k are positive integers. The finiteness of Ramsey numbers are well known [7]. The complementary Ramsey number is defined by replacing ω by α in the above definition of the Ramsey number:

$$\bar{R}(m_1, \dots, m_k) = \min\{n \in \mathbb{N} \mid \forall f \in C(n, k), \exists i \in [k], \alpha_i(f) \geq m_i\}.$$

Clearly,

$$\bar{R}(m_1, \dots, m_k) = \bar{R}(m_{\sigma(1)}, \dots, m_{\sigma(k)})$$

for any permutation σ on $[k]$. The finiteness of the complementary Ramsey number will follow from Lemmas 1 and 3 below.

Lemma 1. *For positive integers m_1, m_2 , we have $R(m_1, m_2) = \bar{R}(m_2, m_1)$.*

Proof. This follows immediately from $\alpha_1(f) = \omega_2(f)$, $\alpha_2(f) = \omega_1(f)$ for any $f \in C(n, 2)$. \square

Lemma 2. *For positive integers m_1, \dots, m_k and m'_1, \dots, m'_k with $m_i \leq m'_i$ for all $i \in [k]$, we have $\bar{R}(m_1, \dots, m_k) \leq \bar{R}(m'_1, \dots, m'_k)$.*

Proof. Immediate from the definition. \square

Lemma 3. *For positive integers m_1, \dots, m_k, m_{k+1} , we have*

$$\bar{R}(m_1, \dots, m_k, m_{k+1}) \leq \bar{R}(m_1, \dots, m_k).$$

Moreover, equality holds if $m_{k+1} \geq \bar{R}(m_1, \dots, m_k)$.

Proof. Let $n = \bar{R}(m_1, \dots, m_k)$ and $g \in C(n, k+1)$. Define $\sigma : [k+1] \rightarrow [k]$ by

$$\sigma(i) = \begin{cases} k & \text{if } i = k+1, \\ i & \text{otherwise.} \end{cases}$$

Set $f = \sigma \circ g \in C(n, k)$. Then there exists $i \in [k]$ such that $\alpha_i(f) \geq m_i$. Since $g^{-1}(i) \subset f^{-1}(i)$, we have $\alpha_i(g) \geq \alpha_i(f) \geq m_i$. Since $g \in C(n, k+1)$ was arbitrary, we conclude $\bar{R}(m_1, \dots, m_{k+1}) \leq n$.

Since $n = \bar{R}(m_1, \dots, m_k)$, there exists $f \in C(n-1, k)$ such that $\alpha_i(f) < m_i$ for any $i \in [k]$. Define a coloring $g \in C(n-1, k+1)$ by

$$g^{-1}(i) = \begin{cases} f^{-1}(i) & \text{if } i \in [k], \\ \emptyset & \text{if } i = k+1. \end{cases}$$

Then $\alpha_i(g) = \alpha_i(f) < m_i$ for $i \in [k]$, and $\alpha_{k+1}(g) = n-1$. Therefore, $\bar{R}(m_1, \dots, m_k, n) > n-1$. If $m_{k+1} \geq n$, then Lemma 2 implies

$$\bar{R}(m_1, \dots, m_k, m_{k+1}) \geq \bar{R}(m_1, \dots, m_k, n) \geq n,$$

and hence equality is forced. \square

By Lemmas 1 and 3, we obtain $\bar{R}(m_1, \dots, m_k) \leq R(m_1, m_2)$. In particular, $\bar{R}(m_1, \dots, m_k)$ is finite.

Lemma 4. For positive integers m_1, \dots, m_k ,

$$\bar{R}(m_1, \dots, m_k, 2) = \min\{m_1, \dots, m_k\}.$$

Proof. Immediate from the definition. \square

We abbreviate

$$\bar{R}(\underbrace{m, m, \dots, m}_k)$$

as $\bar{R}(m; k)$. For a graph H and a positive integer k , $R(H; k)$ denotes the generalized Ramsey number, which is the smallest positive integer n such that for any $f \in C(n, k)$, there exists $i \in [k]$ such that $f^{-1}(i)$ contains a subgraph isomorphic to H .

Lemma 5. For positive integers m and k , let H be any graph obtained from K_m by deleting $k - 2$ edges. Then $\bar{R}(m; k) \leq R(H; k)$.

Proof. Let $n = R(H; k)$ and $f \in C(n, k)$. Then there exists $i \in [n]$ such that $f^{-1}(i)$ contains H as a subgraph. By the assumption, K_m can be reconstructed from H by adding $k - 2$ edges. Let $\{i_1, \dots, i_{k-2}\}$ be the colors of these $k - 2$ edges. Then there exists $j \in [k] \setminus \{i_1, \dots, i_{k-2}\}$, and the vertex set of H is an independent set in the graph $f^{-1}(j)$. Thus $\alpha_j(f) \geq m$. Therefore, $\bar{R}(m; k) \leq n$. \square

Lemma 6. Let k and m be positive integers. Then $\bar{R}(m; k) \geq m$. Moreover, equality holds if and only if $k > \binom{m}{2}$.

Proof. If $k \leq \binom{m}{2}$, then there exists $f \in C(m, k)$ such that $f^{-1}(i) \neq \emptyset$ for all $i \in [k]$. Thus $\alpha_i(f) < m$. This implies $\bar{R}(m; k) > m$. On the other hand, clearly, $k > \binom{m}{2}$ implies $\bar{R}(m; k) = m$. \square

3. GENERAL RESULTS

Let n and k be positive integers. For $f \in C(n, k)$ and $x \in [n]$, set

$$f_i(x) = |\{y \in [n] \mid f(\{x, y\}) = i\}|.$$

Lemma 7. Let n, k, m_1, \dots, m_k and t be positive integers with $1 \leq t \leq k$, and let $f \in C(n, k)$. If

$$\sum_{j=1}^t f_j(x) \geq \bar{R}(m_1, m_2, \dots, m_t, m_{t+1} - 1, \dots, m_k - 1)$$

for some $x \in [n]$, then $\alpha_i(f) \geq m_i$ for some $i \in [k]$.

Proof. Set $Y = \{y \in [n] \mid f(\{x, y\}) \in [t]\}$, so that

$$|Y| = \sum_{j=1}^t f_j(x).$$

Let $g = f|_{\binom{Y}{2}}$. By the assumption, either there exists i with $1 \leq i \leq t$ such that $\alpha_i(g) \geq m_i$, or there exists i with $t < i \leq k$ such that $\alpha_i(g) \geq m_i - 1$.

If $1 \leq i \leq t$, then $\alpha_i(f) \geq \alpha_i(g) \geq m_i$. If $t < i \leq k$, then there exists an independent set Z in $g^{-1}(i)$ with $|Z| = m_i - 1$. Then $Z \cup \{x\}$ is an independent set in $f^{-1}(i)$. This implies $\alpha_i(f) \geq m_i$. \square

Lemma 8. *Let k, m_1, \dots, m_k and t be positive integers. Let $[k] = \bigcup_{j=1}^t M_j$ be a nontrivial partition. For each $i \in [k]$ and $j \in [t]$, define*

$$m_i^{(j)} = \begin{cases} m_i & \text{if } i \in M_j, \\ m_i - 1 & \text{otherwise.} \end{cases}$$

Then

$$\bar{R}(m_1, \dots, m_k) \leq \sum_{j=1}^t \bar{R}(m_1^{(j)}, \dots, m_k^{(j)}) - t + 2.$$

Proof. Let n denote the right-hand side of the inequality. If $f \in C(n, k)$ and $x \in [n]$, then

$$\begin{aligned} \sum_{j=1}^t \sum_{i \in M_j} f_i(x) &= \sum_{i=1}^k f_i(x) \\ &= n - 1 \\ &= \sum_{j=1}^t \bar{R}(m_1^{(j)}, \dots, m_k^{(j)}) - t + 1. \\ &> \sum_{j=1}^t (\bar{R}(m_1^{(j)}, \dots, m_k^{(j)}) - 1). \end{aligned}$$

Thus, there exists $j \in [t]$ such that

$$\sum_{i \in M_j} f_i(x) \geq \bar{R}(m_1^{(j)}, \dots, m_k^{(j)}).$$

By Lemma 7, there exists $i \in [k]$ such that $\alpha_i(f) \geq m_i$. This implies $\bar{R}(m_1, \dots, m_k) \leq n$. \square

A graph G is said to be a Ramsey (s, t) -graph if $\omega(G) < s$ and $\alpha(G) < t$. We write $G \subset H$ if H is a subgraph of G . For a graph G and its subgraph H , we denote the graph $(V(G), E(G) \setminus E(H))$ by $G \setminus H$.

Lemma 9. *Let m_1, m_2, m_3 and n be positive integers greater than 2. Then the following are equivalent.*

- (i) $\bar{R}(m_1, m_2, m_3) \leq n$,
- (ii) *for any two Ramsey (m_3, m_2) -graphs G and H on the vertex set $[n]$ such that $G \supset H$, one has $\alpha(G \setminus H) \geq m_1$.*

Proof. We only prove (ii) implies (i), since a proof of the converse can be easily constructed by reversing the argument. Let $f \in C(n, 3)$, and assume $\alpha_2(f) < m_2$ and $\alpha_3(f) < m_3$. Define graphs G, H by $G = f^{-1}(\{1, 2\})$ and $H = f^{-1}(2)$. Then

$$\begin{aligned} \omega(H) &\leq \omega(G) = \alpha_3(f) < m_3, \\ \alpha(G) &\leq \alpha(H) = \alpha_2(f) < m_2. \end{aligned}$$

Thus both G and H are Ramsey (m_3, m_2) -graphs. By (ii), we have $m_1 \leq \alpha(G \setminus H) = \alpha_1(f)$. Therefore (i) holds. \square

4. FACTORS AND COMPLEMENTARY RAMSEY NUMBERS

A graph G is said to be factorable into spanning subgraphs H_1, H_2, \dots, H_k if its edge-set is a disjoint union of those of H_1, H_2, \dots, H_k . If H is another graph, G is said to be H -factorable if G is factorable into H_1, H_2, \dots, H_k and $H \cong H_i$ for all $i \in [k]$.

Lemma 10. *Let n and k be positive integers. Suppose that the complete graph K_n is factorable into spanning subgraphs H_1, H_2, \dots, H_k . Then*

$$\bar{R}(\alpha(H_1) + 1, \alpha(H_2) + 1, \dots, \alpha(H_k) + 1) > n.$$

Proof. The coloring $f \in C(n, k)$ defined by this factorization satisfies $\alpha_i(f) = \alpha(H_i)$, and this implies the desired inequality. \square

For any two integers m and $n \geq 2$, we can write $m = nq + r$ for unique integers q, r with $0 \leq r < n$. Then

$$\underbrace{(q+1, q+1, \dots, q+1)}_r, \underbrace{(q, q, \dots, q)}_{n-r}$$

is a partition of the integer m . We define a graph $\bar{T}_n(m)$ and an integer $\bar{t}_n(m)$ as follows:

$$\bar{T}_n(m) = rK_{q+1} \cup (n-r)K_q,$$

$$\bar{t}_n(m) = |E(\bar{T}_n(m))| = r \binom{q+1}{2} + (n-r) \binom{q}{2} = \frac{1}{2}q(m-n+r).$$

Clearly

$$(1) \quad \bar{t}_n(m+1) - \bar{t}_n(m) = q.$$

Lemma 11. *Let m and n be positive integers with $n \geq 2$. Let G be a graph of order m . If $|E(G)| < \bar{t}_n(m)$, then $\alpha(G) \geq n+1$. Moreover, if $|E(G)| = \bar{t}_n(m)$ and $\alpha(G) = n$, then $G \cong \bar{T}_n(m)$.*

Proof. See [1, Theorem 8 in IV.2]. \square

Lemma 12. *Let k, m, n be positive integers satisfying $\bar{t}_{m-1}(n) > \frac{1}{k} \binom{n}{2}$. Then $\bar{R}(m; k) \leq n$.*

Proof. Let $f \in C(n, k)$. Then by the assumption, there exists $i \in [k]$ such that $\bar{t}_{m-1}(n) > |f^{-1}(i)|$. Lemma 11 then implies $\alpha_i(f) \geq m$. Thus $\bar{R}(m; k) \leq n$. \square

Theorem 1. *Let k and $N > 1$ be integers. Suppose that the complete graph K_N is factorable into H_1, H_2, \dots, H_k where $H_i \cong r_i K_{q_i+1} \cup (n_i - r_i) K_{q_i}$ for some non-negative integers n_i, q_i, r_i which satisfy $N = n_i q_i + r_i$ and $0 \leq r_i < n_i$ for any $i \in [k]$. Assume further that $(n_i - r_i - 1)q_i > 0$ for some $i \in [k]$. Then*

$$\bar{R}(n_1 + 1, n_2 + 1, \dots, n_k + 1) = N + 1.$$

Proof. By the assumption, we have

$$\begin{aligned}
(N-1) \sum_{i=1}^k q_i &> \sum_{i=1}^k q_i (N - n_i + r_i) \\
&= \sum_{i=1}^k 2\bar{t}_{n_i}(N) \\
&= \sum_{i=1}^k 2|E(H_i)| \\
&= 2|E(K_N)| \\
&= N(N-1).
\end{aligned}$$

Thus

$$(2) \quad \sum_{i=1}^k q_i > N.$$

Since $\bar{t}_{n_i}(N+1) - \bar{t}_{n_i}(N) = q_i$ for any $i \in [k]$, we have

$$\begin{aligned}
\sum_{i=1}^k \bar{t}_{n_i}(N+1) &= \sum_{i=1}^k (\bar{t}_{n_i}(N) + q_i) \\
&= \sum_{i=1}^k |E(H_i)| + \sum_{i=1}^k q_i \\
&> \binom{N}{2} + N \quad (\text{by (2)}) \\
&= |E(K_{N+1})|.
\end{aligned}$$

Therefore, for any $f \in C(N+1, k)$, there exists some $i \in [k]$ such that $\bar{t}_{n_i}(N+1) > |f^{-1}(i)|$. Then $\alpha_i(f) \geq n_i + 1$ by Lemma 11. This implies

$$\bar{R}(n_1 + 1, n_2 + 1, \dots, n_k + 1) \leq N + 1.$$

The reverse inequality follows from Lemma 10. \square

Corollary 1. *Let m, n and r be non-negative integers with $0 \leq r < n$. If K_{mn+r} is $(rK_{m+1} \cup (n-r)K_m)$ -factorable, then*

$$\bar{R}(n+1; N(m, n, r)) = \begin{cases} mn + r + 1 & \text{if } 0 \leq r \leq n-2, \\ mn + n + 1 & \text{if } r = n-1. \end{cases}$$

where

$$N(m, n, r) = \frac{\binom{mn+r}{2}}{r\binom{m+1}{2} + (n-r)\binom{m}{2}}.$$

Proof. If $0 \leq r \leq n-2$, then the result follows from Theorem 1.

Suppose $r = n-1$. Since $K_{(m+1)n-1}$ is $((n-1)K_{m+1} \cup K_m)$ -factorable by the assumption, we see that $K_{(m+1)n}$ is nK_{m+1} -factorable. Since we have already proved the statement for the case $r = 0$, we find $\bar{R}(n+1; N(m+1, n, 0)) = (m+1)n + 1$. Since $N(m+1, n, 0) = N(m, n, n-1)$, the result follows in this case as well. \square

5. EXAMPLES OF FACTORIZATIONS AND CONSEQUENCES

In this section, we give consequences of Corollary 1. It is well known ([3]) that a complete graph of an even order has a 1-factorization. In other words, K_{2n} is nK_2 -factorable. Applying Corollary 1 with $m = 2$ and $r = 0$, we obtain the following corollary.

Corollary 2. *For any integer $n \geq 2$, $\bar{R}(n+1; 2n-1) = 2n+1$.*

Recall that a Kirkman triple system is a resolvable Steiner triple system, or equivalently, nK_3 -factorization of K_{3n} . It is known ([12]) that a Kirkman triple system exists if and only if $3n = 6t+3$ for some non-negative integer t . Applying Corollary 1 with $m = 3$ and $r = 0$, we obtain the following corollary.

Corollary 3. *For any non-negative integer t , $\bar{R}(2t+2; 3t+1) = 6t+4$.*

A resolvable $2-(n, 4, 1)$ design is an nK_4 -factorization of K_{4n} . It is known ([12]) that such a design exists if and only if $4n = 12t+4$ for some non-negative integer t . Applying Corollary 1 with $m = 4$ and $r = 0$, we obtain the following corollary.

Corollary 4. *For any non-negative integer t , $\bar{R}(3t+2; 4t+1) = 12t+5$.*

In design theory, $(pK_m \cup qK_{m+1})$ -factorization of K_v is called a class-uniform resolvable design $\text{CURD}(v, k, p, q)$ (of index 1), where

$$k = \frac{\binom{v}{2}}{p\binom{m}{2} + q\binom{m+1}{2}}.$$

Class-uniform resolvable designs are investigated in several papers (see, for example, [4]).

Corollary 5. *If there exists a $\text{CURD}(v, k, p, q)$, then $\bar{R}(p+q+1; k) = v+1$.*

A complete set of mutually orthogonal Latin squares (MOLS) of order n can be identified with an edge-coloring $f : E(K_{n^2}) \rightarrow [n+1]$ such that $f^{-1}(i) \cong nK_n$ for all $i \in [n+1]$.

Theorem 2. *There exists a complete set of MOLS of order n if and only if $\bar{R}(n+1; n+1) = n^2+1$.*

Proof. If there exists a complete set of MOLS of order n , then by Corollary 1, we have $\bar{R}(n+1; n+1) = n^2+1$.

Conversely, suppose $\bar{R}(n+1; n+1) = n^2+1$. In particular, $\bar{R}(n+1; n+1) > n^2$. This means that there exists $f \in C(n^2; n+1)$ such that $\alpha_i(f) \leq n$ for any $i \in [n+1]$. Then by Lemma 11, we have $|f^{-1}(i)| \geq \bar{t}_n(n^2)$. Since

$$\frac{1}{n+1} \binom{n^2}{2} = \bar{t}_n(n^2),$$

equality holds in the above inequality, and we have $f^{-1}(i) = \bar{T}_n(n^2) = nK_n$ for any $i \in [n+1]$ by the second part of Lemma 11. \square

Table 1 lists the values of N, n, m, r, k such that K_N is $(rK_{m+1} \cup (n-r)K_m)$ -factorable into k subgraphs. The families I, II, III and IV correspond to Corollaries 2, 3, 4 and Theorem 2, respectively. Note that affine planes which give the family IV in Table 1 are known to exist when q is a prime power ([12]).

	N	n	m	r	k	$R(n+1; k)$	references
I	$2t$	t	2	0	$2t-1$	$2t+1$	1-factor
II	$6t+3$	$2t+1$	3	0	$3t+1$	$6t+4$	[12]
III	$12t+4$	$3t+1$	4	0	$4t+1$	$12t+5$	[12]
IV	q^2	q	q	0	$q+1$	q^2+1	affine plane
V	$12t+9$	$5t+4$	2	$3t+3$	$8t+6$	$12t+10$	[4]
VI	$36t+9$	$16t+4$	2	$12t+3$	$27t+6$	$36t+10$	[4]

TABLE 1. $\bar{R}(n+1; k)$ determined by factorizations

Many other factorizations of K_n satisfying the assumption of Theorem 1 are known in terms of designs. A nearly Kirkman triple system is a factorization of K_{6n} with $H_i \cong 2nK_3$ for $i \in [k-1]$ and $H_k \cong 3nK_2$. This exists if and only if $n \geq 3$. A uniformly resolvable pairwise balanced design is a factorization of K_n satisfying Theorem 1 with $r_i = 0$ for any $i \in [k]$. For the cases $m_i \in \{1, 2\}$, these designs exist if and only if $n = 6t$ and $3t \leq k \leq 6t-1$ for a non-negative integer t with two exceptions, corresponding to the non-existence of nearly Kirkman triple systems of order 6 and 12 ([9]). A restricted resolvable design $R_m RP(n, k)$ is a factorization of K_n with $H_i = pK_m \cup qK_{m+1}$ for some non-negative integer p and q . For the case $m = 2$, an additional construction of factorizations and the existence problems of such factorizations are discussed in [9], [10]. Moreover, a class-uniformly resolvable group divisible design of type g^a with partition $s^p t^q$ with k resolution classes is a factorization of K_{g^a} with $H_i \cong pK_s \cup qK_t$ for any $i \in [k-1]$ and $H_k \cong aK_g$. For the case $s = 2$ and $t = 3$, a number of infinite families of factorizations are constructed in [5], [6].

6. $\bar{R}(m_1, m_2, m_3)$ FOR SMALL m_1, m_2, m_3

Lemma 13. *Let G be a 3-regular graph of order 10 and assume that G does not contain K_4 . Then $\alpha(G) \geq 4$.*

Proof. By the theorem of Brooks (see [1, Theorem 3 in V.1]), G is 3-colorable. Then one of the color class has size greater than $10/3$. This implies that G has an independent set of size at least 4. \square

We remark that the statement of Lemma 13 does not hold when G contains K_4 . This is because the disjoint union of K_4 and the 3-prism is a 3-regular graph of order 10 with independence number 3.

Lemma 14. *Let G be a triangle-free graph of order 8 with a vertex of degree at most one. Then $\alpha(G) \geq 4$.*

Proof. Suppose that 8 is a vertex of degree at most one. Then we may assume i and 8 are not adjacent for any $i \in [6]$. Let H be the graph induced by $[6]$. Since H is triangle-free and $R(3, 3) = 6$, there exists an independent set W of H with $|W| \geq 3$. Then $W \cup \{8\}$ is an independent set of G . Therefore $\alpha(G) \geq 4$. \square

Theorem 3. (i) $\bar{R}(3, 3, 3) = \bar{R}(4, 3, 3) = \bar{R}(5, 3, 3) = 5$, $\bar{R}(6, 3, 3) = 6$.
(ii) $\bar{R}(4, 4, 3) = 7$, $\bar{R}(5, 4, 3) = \bar{R}(6, 4, 3) = 8$, $\bar{R}(7, 4, 3) = 9$.
(iii) $\bar{R}(4, 4, 4) = 10$.

Proof. (i) First we note that $\bar{R}(3, 3, 3) = 5$ follows from Corollary 2.

To show $\bar{R}(4, 3, 3) = \bar{R}(5, 3, 3) = 5$, it is enough to show $\bar{R}(5, 3, 3) \leq 5$ by Lemma 2. Suppose that $f \in C(5, 3)$ satisfies $\alpha_2(f) \leq 2$ and $\alpha_3(f) \leq 2$. Since C_5 is a unique simple graph G of order 5 which satisfy $\alpha(G) \leq 2$ and $\omega(G) \leq 2$, both $f^{-1}(2)$ and $f^{-1}(3)$ are C_5 . This forces $f^{-1}(1) = \emptyset$ and hence $\alpha_1(f) = 5$. Therefore $\bar{R}(5, 3, 3) \leq 5$.

Finally we show $\bar{R}(6, 3, 3) = 6$. Since $\bar{R}(3, 3) = R(3, 3) = 6$, this follows from Lemma 3.

(ii) First we show $\bar{R}(4, 4, 3) = 7$. Since $E(K_6)$ can be partitioned into one $E(\bar{C}_6)$ and two $E(3K_2)$'s, $\alpha(\bar{C}_6) = 2$, and $\alpha(3K_2) = 3$, we obtain $\bar{R}(4, 4, 3) > 6$. To prove the reverse inequality, suppose that there exists $f \in C(7, 3)$ satisfying $\alpha_1(f) \leq 3$, $\alpha_2(f) \leq 3$ and $\alpha_3(f) \leq 2$. Then by Lemma 7 and (i), we have $f_3(x) \leq \bar{R}(3, 3, 3) - 1 = 4$. On the other hand, by Lemmas 7 and 4, we have $f_3(x) = 6 - (f_1(x) + f_2(x)) \geq 6 - (\bar{R}(4, 4, 2) - 1) = 3$ for any $x \in [7]$. Since there is no 3-regular graph on 7 vertices, we may assume $f_3(1) = 4$. Then we may assume $f(\{1, i\}) = 3$ for $i \in Y = \{2, 3, 4, 5\}$. Setting $g = f|_{\binom{Y}{2}}$, we have $\alpha_i(g) \leq 2$ for any $i \in [3]$. This implies that g is a 1-factorization of K_4 . Without loss of generality, we may assume that $g(\{2, 3\}) = g(\{4, 5\}) = 1$, $g(\{2, 4\}) = g(\{3, 5\}) = 2$ and $g(\{2, 5\}) = g(\{3, 4\}) = 3$. Moreover we may assume $f(\{1, 6\}) = 1$. Since $\{1, 2, 5, 6\}$ cannot be an independent set in $f^{-1}(2)$, we may assume $f(\{2, 6\}) = 2$. Since $\{2, 3, 6\}$ and $\{2, 4, 6\}$ cannot be independence sets in $f^{-1}(3)$, we have $f(\{3, 6\}) = f(\{4, 6\}) = 3$. Then $\{1, 3, 4, 6\}$ is an independence set in $f^{-1}(2)$, contradicting $\alpha_2(f) \leq 3$. Therefore $\bar{R}(4, 4, 3) \leq 7$.

Next we show $\bar{R}(5, 4, 3) = \bar{R}(6, 4, 3) = 8$. Define a coloring $f \in C(7, 3)$ by

$$\begin{aligned} f^{-1}(1) &= \{\{2, 5\}, \{3, 6\}, \{4, 7\}\}, \\ f^{-1}(2) &= \{\{1, 2\}, \{2, 3\}, \dots, \{7, 1\}\} \cong C_7, \\ f^{-1}(3) &= \binom{[7]}{2} \setminus (f^{-1}(1) \cup f^{-1}(2)). \end{aligned}$$

Then $\alpha_1(f) = 4$, $\alpha_2(f) = 3$ and $\alpha_3(f) = 2$. Therefore $7 < \bar{R}(5, 4, 3) \leq \bar{R}(6, 4, 3)$. To prove $\bar{R}(5, 4, 3) = \bar{R}(6, 4, 3) = 8$, we show $\bar{R}(6, 4, 3) \leq 8$. Suppose that there exists $f \in C(8, 3)$ such that $\alpha_1(f) \leq 5$, $\alpha_2(f) \leq 3$ and $\alpha_3(f) \leq 2$. By Lemma 4, Lemma 7 and (i),

$$(3) \quad f_3(i) \leq \bar{R}(5, 3, 3) - 1 = 4,$$

$$(4) \quad f_1(i) + f_2(i) \leq \bar{R}(6, 4, 2) - 1 = 3,$$

for any $i \in [8]$. Since $f_1(i) + f_2(i) + f_3(i) = 7$, equality holds in (3) and (4). Since $\alpha_3(f) \leq 2$, $f^{-1}(2)$ is triangle-free. Since $\alpha_2(f) \leq 3$, Lemma 14 implies $f_2(i) \geq 2$ for all $i \in [8]$. Since $\alpha_1(f) \leq 5$, $f^{-1}(1) \neq \emptyset$, so by (4), there exists $i \in [8]$ such that $f_2(i) = 2$. Therefore we may assume $f_2(8) = 2$ and $f(\{6, 8\}) = f(\{7, 8\}) = 2$. Since $\alpha_3(f) \leq 2$, we have $f(\{6, 7\}) = 3$. Let g denote the restriction of f to $\binom{[5]}{2}$. Then by the assumption $\alpha_3(g) \leq \alpha_3(f) \leq 2$. Moreover, if Y is an independence set of $g^{-1}(2)$, then $Y \cup \{8\}$ is an independence set of $f^{-1}(2)$. Therefore $\alpha_2(g) \leq \alpha_2(f) - 1 \leq 2$. This implies $g^{-1}(2) \cong g^{-1}(3) \cong C_5$ and $\binom{[5]}{2} \cap f^{-1}(1) = \emptyset$. We may assume $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}\} \subset f^{-1}(2)$ and $\{\{1, 3\}, \{3, 5\}, \{2, 5\}, \{2, 4\}, \{1, 4\}\} \subset f^{-1}(3)$. Since $f_2(i) \geq 2$ and $f_1(i) \leq 1$ for each $i \in [8]$ and $\alpha_1(f) \leq 5$, each of $\{6, 7, 8\}$ is adjacent in $f^{-1}(1)$ to a unique vertex in $[5]$.

By symmetry, we may assume $f(\{1, 8\}) = 1$. Now since $f_1(1) = 1$ and $f_2(1) = 2$, we can apply the same argument as above, using 1, $\{1, 2\}$ and $\{1, 5\}$ instead of 8, $\{6, 8\}$ and $\{7, 8\}$, respectively. Then we obtain $\binom{\{3, 4, 6, 7, 8\}}{2} \cap f^{-1}(1) = \emptyset$. In particular, neither 6 nor 7 is adjacent in $f^{-1}(1)$ to 3 or 4. Then by symmetry, we may assume $f(\{2, 6\}) = f(\{5, 7\}) = 1$. Since $f(\{1, 2\}) = f(\{2, 3\}) = 2$, we can again apply the same argument, using 2, $\{1, 2\}$ and $\{2, 3\}$, and obtain $\binom{\{4, 5, 6, 7, 8\}}{2} \cap f^{-1}(1) = \emptyset$. However, this contradicts $f(\{5, 7\}) = 1$.

Finally, we show $\bar{R}(7, 4, 3) = 9$. Define a coloring $f \in C(8, 3)$ by

$$\begin{aligned} f^{-1}(1) &= \{\{1, 5\}, \{2, 6\}\}, \\ f^{-1}(2) &= \{\{1, 2\}, \{2, 3\}, \dots, \{8, 1\}\} \cup \{\{3, 7\}, \{4, 8\}\}, \\ f^{-1}(3) &= \binom{[8]}{2} \setminus (f^{-1}(1) \cup f^{-1}(2)). \end{aligned}$$

Then $\alpha_1(f) = 6$, $\alpha_2(f) = 3$ and $\alpha_3(f) = 2$. Therefore $\bar{R}(7, 4, 3) > 8$. On the other hand, $\bar{R}(k, 4, 3) \leq \bar{R}(4, 3) = R(4, 3) = 9$. Therefore $\bar{R}(k, 4, 3) = 9$ for $k \geq 7$.

(iii) Since $\bar{R}(4, 4, 4) \geq \bar{R}(4, 4, 4, 4)$ by Lemma 3, we see $\bar{R}(4, 4, 4) \geq 3^2 + 1 = 10$ by Theorem 2. To prove the reverse inequality, let $f \in C(10, 3)$ and $x \in [10]$. If $f_i(x) + f_j(x) \geq \bar{R}(4, 4, 3)$ for some distinct $i, j \in [3]$, then $\alpha_k(f) \geq 4$ for some $k \in [3]$ by Lemma 7. Thus we may assume $f_i(x) + f_j(x) \leq \bar{R}(4, 4, 3) - 1$ for any distinct $i, j \in [3]$. Then

$$9 = f_1(x) + f_2(x) + f_3(x) \leq \frac{3}{2}(\bar{R}(4, 4, 3) - 1) = 9$$

by (ii), hence equality is forced. This implies that $f_i(x) = 3$ for any $i \in [3]$. If $f^{-1}(3)$ contains K_4 , then $\alpha_1(f) \geq 4$. If $f^{-1}(3)$ does not contain K_4 , then by Lemma 13, we have $\alpha_3(f) \geq 4$. Therefore $\bar{R}(4, 4, 4) \leq 10$. \square

Theorem 4. (i) $\bar{R}(4; 6) = 5$,
(ii) $\bar{R}(5; k) = 6$ for $8 \leq k \leq 10$,
(iii) $\bar{R}(6; 8) = 11$, $\bar{R}(6; 10) = 8$ and $\bar{R}(6; k) = 7$ for $11 \leq k \leq 15$.

Proof. Note that the upper bounds for each of these complementary Ramsey numbers are derived from Lemma 12. Note also that the lower bounds are derived from Lemma 6 for $\bar{R}(4; 6)$, $\bar{R}(5; k)$ for $8 \leq k \leq 10$, and $\bar{R}(6; k)$ for $11 \leq k \leq 15$. Also, $\bar{R}(6; 8) \geq \bar{R}(6; 9) = 11$ by Lemma 3 and Corollary 2. Thus, it remains to show $\bar{R}(6; 10) \geq 8$.

Define $f \in C(7, 10)$ by

$$\begin{aligned} f^{-1}(i) &= \{\{i+2, i+5\}, \{i+3, i+4\}\} \text{ if } i \in [7], \\ f^{-1}(8) &= \{\{1, 3\}, \{2, 4\}\}, \\ f^{-1}(9) &= \{\{3, 5\}, \{4, 6\}\}, \\ f^{-1}(10) &= \{\{5, 7\}, \{6, 1\}, \{2, 7\}\}, \end{aligned}$$

where, in the definition of $f^{-1}(i)$, the elements are read modulo 7. Then $\alpha_i(f) = 5$ for any $i \in [10]$, and hence $\bar{R}(6; 10) > 7$. \square

In Table 2, the values with subscripts I–V are given by the corresponding row of Table 1, those with subscript b are given by Lemma 6, while those with c are given by Theorem 4. We remark that upper bounds for some of the complementary

k	2	3	4	5	6	7	8	9	10	11	\cdots	15	16
$R(3; k)$	6	5_I	3_b	\cdots									
$R(4; k)$	18	10	10_{II}	7_I	5_c	4_b	\cdots						
$R(5; k)$?	?	?	17_{III}	10_V	9_I	6_c	6_c	6_c	5_b	\cdots		
$R(6; k)$?	?	?	?	26_{IV}	16_{II}	11_c	11_I	8_c	7_c	\cdots	7_c	6_b

TABLE 2. $\bar{R}(m; k)$ for $m = 3, 4, 5, 6$

m_1	3	4	5	6	7	8
$R(m_1, 3, 3)$	5	5	5	6	\cdots	
$R(m_1, 4, 3)$	-	7	8	8	9	\cdots

TABLE 3. $\bar{R}(m_1, m_2, 3)$ for $m_2 = 3, 4$

Ramsey numbers in Table 3 can also be derived from known list of Ramsey (s, t) -graphs, using Lemma 9.

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